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Some properties of the *k*-dimensional Lyness' map

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Abstract

This paper is devoted to study some properties of the k-dimensional Lyness' map $F(x_1,\ldots,x_k)=\left(x_2,\ldots,x_k,\left(a+\sum_{i=2}^{k}x_i\right)\big/x_1\right)$. Our main result presents a rational vector field that gives a Lie symmetry for F. This vector field is used, for $k\leqslant 5$, to give information about the nature of the invariant sets under F. When k is odd, we also present a new (as far as we know) first integral for $F\circ F$ which allows us to deduce in a very simple way several properties of the dynamical system generated by F. In particular for this case we prove that, except on a given codimension one algebraic set, none of the positive initial conditions can be a periodic point of odd period.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction and main results

Discrete integrable systems are the focus of current intensive research (see [14, 18, 23, 26, 34, 35] and references therein) since they appear as fundamental mathematical tools in numerical analysis and in some areas of physics and theoretical biology such as statistical mechanics [28, 36], discrete quantum theory [5, 6], solitons in cellular automata [22, 27, 33] and population dynamics [32] among other topics.

There are some well-known planar integrable maps like the Lyness and McMillan ones, generalized by the celebrated QRT maps [29, 30], and recently a list of third-order integrable difference equation (including the third-order Lyness one) has attracted the researchers attention [18, 26, 31]. In this context, the second- and the third-order Lyness' difference equations

$$y_{n+2}=\frac{a+y_{n+1}}{y_n}\qquad\text{and}\qquad y_{n+3}=\frac{a+y_{n+1}+y_{n+2}}{y_n},\qquad\text{with}\quad a\geqslant 0$$

have been considered as paradigmatic examples of integrable discrete systems. The dynamics of the above equations, or their associated maps, has been the objective of recent intensive investigation. Nowadays, for these cases, the behavior of the orbits is well known when positive initial conditions are considered (see [2, 7, 9, 12, 37]), although few results have been obtained for negative initial conditions, see [8, 12]. The development of techniques to study the dynamics of the second- and third-order Lyness' equations have been the starting point to study wide classes of integrable systems see for instance [3, 10, 12, 16, 19, 20, 21, 31], and [11] for a general paper on this topic.

For $k \geqslant 4$, very few results, apart from those obtained recently by Bastien and Rogalski [4], are known for the kth-order Lyness' equation

$$y_{n+k} = \frac{a + \sum_{i=1}^{k-1} y_{n+i}}{y_n}. \tag{1}$$

The main difference between the $k = 2, 3$ and the $k \geqslant 4$ scenarios is that the first cases are *integrable* in the sense that the associated maps have 1 and 2 functionally independent first integrals, respectively (in this paper, we say that a map F is integrable if it has $k-1$ functionally independent first integrals). It seems that this property is not shared for the Lyness' equations when $k \geqslant 4$.

One geometrical object that has played a key role to understand the dynamics of a large class of integrable two- and three-dimensional maps is the *Lie symmetry* of the map [9, 10]. Although for $k \geqslant 4$ it seems that equation (1) is not integrable, we will prove that its associated map still has a Lie Symmetry. The main goal of the paper is to find this symmetry and take advantage of it to study the dynamics of the map associated with (1).

Consider the k-dimensional Lyness' map associated with the difference equation (1),

$$F(x_1, \dots, x_k) = \left(x_2, \dots, x_k, \frac{a + \sum_{i=2}^{k} x_i}{x_1}\right), \qquad \text{with} \quad a \geqslant 0, \tag{2}$$

which is a diffeomorphism from $Q^+ := \{\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k : x_1 > 0, x_2 > 0, \dots, x_k > 0\}$ into itself. It is well known that it has the following couple of functionally independent first integrals:

$$V_1(\mathbf{x}) = \left(a + \sum_{i=1}^{k} x_i\right)\left(\prod_{i=1}^{k}(x_i + 1)\right) \Big/ (x_1 \cdots x_k) \tag{3}$$

and

$$V_2(\mathbf{x}) = \left(a + \sum_{i=1}^{k} x_i + x_1 x_k\right)\left(\prod_{i=1}^{k-1}(1 + x_i + x_{i+1})\right) \Big/ (x_1 \cdots x_k). \tag{4}$$

A third functionally independent first integral for $k \geqslant 5$ has recently been given in [13]. Moreover in that paper it is conjectured that for any k, the Lyness' map has up to $E\left(\frac{k+1}{2}\right)$ functionally independent first integrals, where $E(\cdot)$ denotes the integer part function. The conjecture seems to be true for $k \leqslant 6$, see again [13].

The integrable structure for $k = 2, 3$ implies that the dynamics of the maps studied in the above references is in fact one dimensional. In any case, although for $k \geqslant 4$ the above assertion seems not to be true, the existence of several first integrals reduces the dimension of the space where the dynamics takes place. If the above conjecture would be true, then $k - E\left(\frac{k+1}{2}\right)$ would be generically the dimension of the invariant manifold given by the level sets of the first integrals.

Recall that a vector field $\mathbf{X}$ is said to be a Lie symmetry of a map G if it satisfies the condition

$$\mathbf{X}(G(\mathbf{x})) = (DG(\mathbf{x}))\mathbf{X}(\mathbf{x}). \tag{5}$$

The vector field $\mathbf{X}$ is related to the dynamics of G in the following sense: G maps any orbit of the differential system determined by the vector field, to another orbit of this system, see [10]. In the integrable case, where the dynamics are in fact one dimensional, the existence of a Lie symmetry fully characterizes the dynamics. In [10, theorem 1] it is proved that if $G : \mathcal{U} \to \mathcal{U}$ is a diffeomorphism having a Lie symmetry $\mathbf{X}$, and such G preserves γ, a solution of the differential equation $\dot{x} = \mathbf{X}(x)$, then the dynamics of $G|_\gamma$ is either conjugated to a rotation, conjugated to a translation of the line, or constant, according whether γ is homeomorphic to $\mathbb{S}^1$, $\mathbb{R}$, or a point, respectively. Other properties of the Lie symmetries of discrete systems are studied in [17].

One of the main results of this paper is the following theorem where the expression of a Lie symmetry for the k-dimensional Lyness' map is given.

Theorem 1. *For $k \geqslant 3$, the vector field $\mathbf{X}_k = \sum_{i=i}^k X_i \frac{\partial}{\partial x_i}$ is a Lie symmetry for the k-dimensional Lyness' map (2), where*

$$X_1(\mathbf{x}) = \frac{(x_1+1)\left[\prod_{i=2}^{k-1}(1+x_i+x_{i+1})\right]\left(a+\sum_{i=1}^{k-1}x_i - x_2x_k\right)}{\prod_{i=2}^k x_i}, \tag{6}$$

$$X_m(\mathbf{x}) = \frac{(x_m+1)\left[\prod_{i=1, i\neq m-1,m}^{k-1}(1+x_i+x_{i+1})\right]\left(a+\sum_{i=1}^{k}x_i + x_1x_k\right)(x_{m-1}-x_{m+1})}{\prod_{i=1,i\neq m}^k x_i}, \tag{7}$$

for all $2 \leqslant m \leqslant k-1$ and

$$X_k(\mathbf{x}) = -\frac{(x_k+1)\left[\prod_{i=1}^{k-2}(1+x_i+x_{i+1})\right]\left(a+\sum_{i=2}^{k}x_i - x_1x_{k-1}\right)}{\prod_{i=1}^{k-1} x_i}. \tag{8}$$

Once we have the candidate $\mathbf{X}_k$ to be a Lie symmetry of the Lyness' map F the proof of theorem 1 only will consist in checking that (5) holds. We give here some hints of how we have found the above $\mathbf{X}_k$. Observe that if there exists a vector field $\mathbf{X}_k$, satisfying equation (5) for the k-dimensional Lyness' map (2), then

$$\begin{pmatrix} X_1(F) \\ X_2(F) \\ \vdots \\ X_k(F) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ -\dfrac{a+\sum_{i=2}^k x_i}{x_1^2} & \dfrac{1}{x_1} & & & \cdots & \dfrac{1}{x_1} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{pmatrix}.$$

Hence it is necessary that

$$X_{i+1} = X_i(F), \qquad \text{for} \quad i = 1, \ldots, k-1, \tag{9}$$

and the *'compatibility condition'*

$$X_k(F) = -\left(\frac{a+\sum_{i=2}^k x_i}{x_1^2}\right)X_1 + \frac{1}{x_1}\left[\sum_{i=2}^k X_i\right].$$

Thus, the construction of a Lie symmetry is straightforward once the right expression of X_1, as a seed of equations (9), is obtained. In [7, 9] and [10] the expressions of $\mathbf{X}_2$ and $\mathbf{X}_3$ are given respectively. The idea of these papers is that these vector fields have to be multiples of ∇V_1 and $\nabla V_1 \times \nabla V_2$, respectively. These constructions are used to force F and $\mathbf{X}_k$ to share the same set of first integrals. When $k \geqslant 4$ we cannot use anymore this idea because there are no enough first integrals. Nevertheless we observe that the first components of $\mathbf{X}_2$ and $\mathbf{X}_3$ are

$$\frac{(x_1+1)\left(a+x_1-x_2^2\right)}{x_2} \qquad \text{and} \qquad \frac{(x_1+1)(1+x_2+x_3)(a+x_1+x_2-x_2x_3)}{x_2x_3},$$

respectively. Thus it seems natural to try with X_1 as the expression given in (6) and, indeed, it works! From this starting point, the proof for a given small k is only a matter of computations, while the proof for a general k is long and tedious, but straightforward. It is done in section 2. We suggest to skip this section in a first reading of the paper.

Another result that helps for understanding the dynamics generated by F, when k is odd, is given in following proposition. In this result, the key point is the existence of a new (as far as we know) first integral for $F^2 = F \circ F$ for any odd $k \geqslant 3$. As we will see, our proof of the existence of this function is inspired in the paper [26], where this first integral is given for $k = 3$.

Theorem 2. *Set $k = 2\ell + 1$, $\mathbf{x} = (x_1, \ldots, x_{2\ell+1})$ and consider F from $\mathcal{Q}^+$ into itself. Then*

(a) The function

$$W(\mathbf{x}) = \frac{\prod_{j=0}^{\ell}(x_{2j+1}+1)}{\prod_{j=1}^{\ell} x_{2j}} \tag{10}$$

is a first integral of F^2.

(b) For any $\ell \geqslant 2$, the function $V_3 := W + W(F)$, which is

$$V_3(\mathbf{x}) = \frac{\prod_{j=0}^{\ell} x_{2j+1}(x_{2j+1}+1) + \left(a + \sum_{j=1}^{2\ell+1} x_j\right)\prod_{j=1}^{\ell} x_{2j}(x_{2j}+1)}{\prod_{i=1}^{2\ell+1} x_i}, \tag{11}$$

is a first integral of F which is functionally independent with the first integrals V_1 and V_2 given in (3) and (4), respectively.

(c) The function $W \cdot W(F)$ coincides with the first integral of F, V_1 given in (3). In other words, $V_1 = W \cdot W(F)$.

(d) The algebraic set $\mathcal{G} := \{\mathbf{x} \in \mathcal{Q}^+ : W(\mathbf{x}) - W(F(\mathbf{x})) = 0\}$ is invariant by F.

(e) If the map F has some periodic point of odd period then it has to be contained in $\mathcal{G}$.

(f) Setting $\mathcal{G}^{\pm} := \{\mathbf{x} \in \mathcal{Q}^+ : \pm(W(\mathbf{x}) - W(F(\mathbf{x}))) > 0\}$, the map F sends $\mathcal{G}^+$ into $\mathcal{G}^-$ and vice versa, and both sets are invariant by F^2. Furthermore, the dynamics of F^2 on each of these sets are conjugated, being the map F itself the conjugation.

(g) The measure

$$m_1(B) := \int_B \frac{\pm 1}{\Pi(\mathbf{x})(W(\mathbf{x}) - W(F(\mathbf{x})))}\, \mathrm{d}\mathbf{x},$$

where $\Pi(\mathbf{x}) = \prod_{i=1}^{k} x_i$, is an invariant measure for F^2, i.e. $m_1(F^2(B)) = m_1(B)$, where B is any measurable set in $\mathcal{G}^{\pm}$.

(h) The measure

$$m_2(B) := \int_B \frac{1}{\Pi(\mathbf{x})}\, \mathrm{d}\mathbf{x}$$

is an invariant measure for F^2, i.e. $m_2(F^2(B)) = m_2(B)$, where B is any measurable set in $\mathcal{Q}^+$.

We remark that, when k is odd, the first integral V_3 given above coincides with the one given recently in [13]. Observe also that the invariant algebraic surface $\mathcal{G}$ was already found in [9], but only for $k = 3$. Also, as we will see in subsection 3.1, the function W is useful to make an explicit simple-order reduction when we study the dynamics of F for k odd.

Although, by using both theorems we have not been able to present a complete study of the higher dimensional Lyness' map, in next results we give some information about the invariant sets in the phase space when $k = 4, 5$. We prove:

Proposition 3. *The vector field* $\mathbf{X}_4$ *given by equations (6)–(8) for* $k = 4$ *is a Lie symmetry for the four-dimensional Lyness' map. Moreover,* $\mathbf{X}_4(V_i) = 0$*, for* $i = 1, 2$*, and then the sets* $I_{h,k} := \{V_1 = h\} \cap \{V_2 = k\} \cap \mathcal{Q}^+$ *are invariant by* F *and by the flow of* $\mathbf{X}_4$.

Furthermore, if we assume that both first integrals intersect transversally on $C_{h,k}$*, a connected component of* $I_{h,k}$*, then* $C_{h,k}$ *is diffeomorphic to a torus.*

Proposition 4. *The vector field* $\mathbf{X}_5$ *given by equations (6)–(8) for* $k = 5$ *is a Lie symmetry for the five-dimensional Lyness' map. Moreover,* $\mathbf{X}_5(V_i) = 0$*, for* $i = 1, 2, 3$*, where*

$$V_3(\mathbf{x}) = \frac{1}{x_1x_2x_3x_4x_5}(x_1x_3x_5(1+x_1)(1+x_3)(1+x_5) + x_2x_4(1+x_2)(1+x_4)(a+x_1+x_2+x_3+x_4+x_5)),$$

and the sets $I_{h,k,l} := \{V_1 = h\} \cap \{V_2 = k\} \cap \{V_3 = \ell\} \cap \mathcal{Q}^+$*, which generically have at least two connected components, are invariant by* F *and by the flow of* $\mathbf{X}_5$.

Furthermore, if we assume that the three first integrals intersect transversally on $C_{h,k,\ell}$*, a connected component of* $I_{h,k,\ell}$*, then* $C_{h,k,\ell}$ *is diffeomorphic to a (two-dimensional) torus.*

It is important to note that in the above results we do not assert that most of the connected components $C_{h,k}$ and $C_{h,k,\ell}$ are tori, since we did not succeed to prove that over them the intersection of the energy levels at the whole sets $C_{h,k}$ and $C_{h,k,\ell}$ are transversal. Hence it remains open to decide whether they are two-dimensional differentiable manifolds or not, and in the case that they are not differentiable manifolds to decide which are their topology. In any case, our result reduces the problem to a computational question.

Our numerical simulations seem to indicate that for $k = 4$, all the generic level curves $I_{h,k}$ are connected. On the other hand, for $k = 5$, generically the sets $I_{h,k,l}$ seem to have exactly two connected components. In figures 1 and 2 we give a projection in $\mathbb{R}^3$ of these surfaces. Indeed, in figure 1 we represent both an orbit of F and an orbit of $\mathbf{X}_4$ starting with the same initial condition, and in figure 2 an orbit of F for $k = 5$. Note that in both cases the behavior of the orbits seems to indicate that two (respectively three) is the maximum number of independent first integrals for F when $k = 4$ (resp. $k = 5$), as it is suggested in [13].

A remarkable fact that figure 1 shows is that, although for $k = 4$ the manifold $I_{h,k}$ is invariant for both the map F and the flow of $\mathbf{X}_4$, the map F seems to send an orbit of $\mathbf{X}_4$ to a different orbit of the vector field. Further numerical experiments seem to confirm this fact. Under this situation, we cannot use the techniques developed in [10]. This fact makes more difficult the knowledge of the behavior of F restricted to each $I_{h,k}$ and is one of the important differences between the cases $k = 2, 3$ and 4.

2. Proof of theorem 1

As we have already explained in the introduction, the existence of a vector field $\mathbf{X}_k = \sum_{i=i}^{k} X_i \frac{\partial}{\partial x_i}$ satisfying equation (5) for the k-dimensional Lyness' map (2) is equivalent to the set of equations

$$X_{i+1} = X_i(F), \qquad \text{for} \quad i = 1, \ldots, k-1, \tag{12}$$

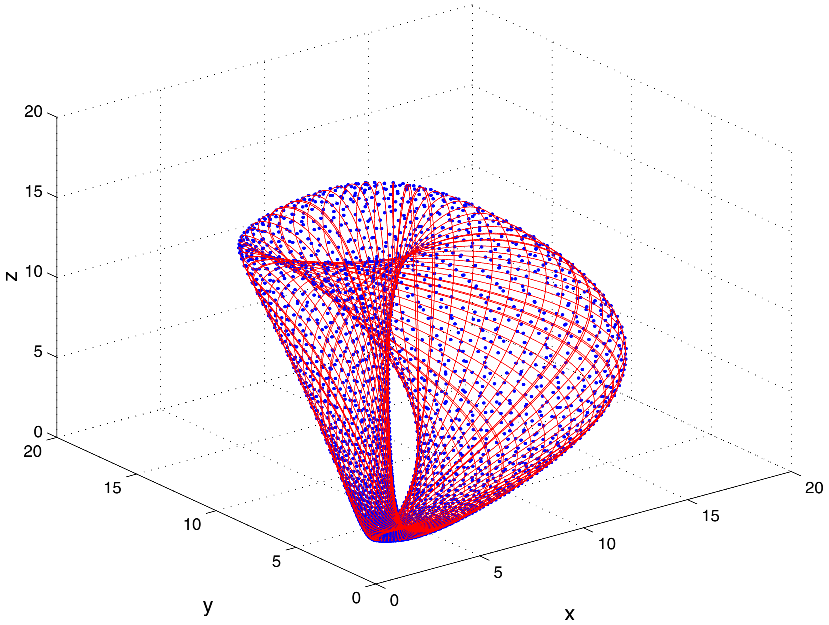

Figure 1. Projections into $\mathbb{R}^3$ of the flow associated with the Lie symmetry $\mathbf{X}_4$, and the orbit of the Lyness' map, for $k = 4$ and $a = 4$, both with initial condition (1, 2, 3, 4).

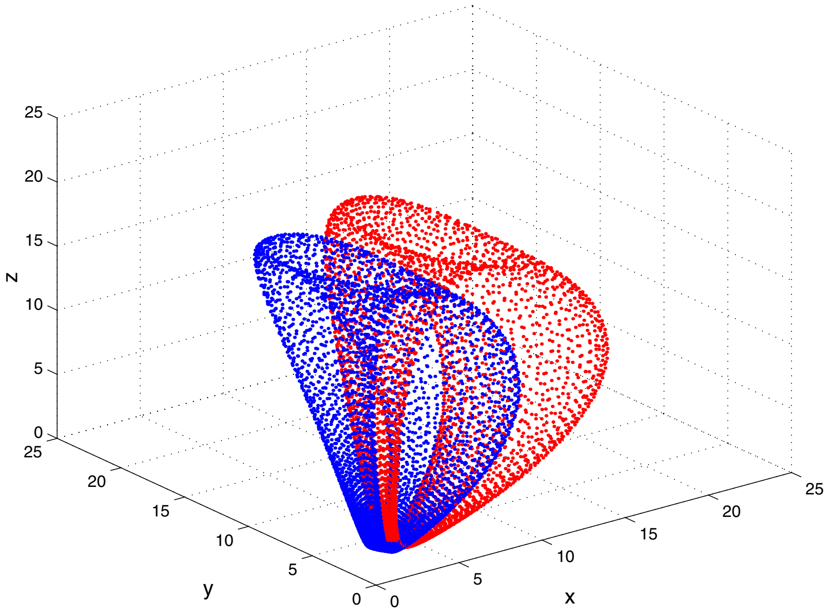

Figure 2. Projection into $\mathbb{R}^3$ of the first 5000 iterates of the Lyness' map for $k = 5$ and $a = 1$, staring at (1, 2, 3, 4, 5). Odd and even iterates are in different connected components.

together with the compatibility condition

$$X_k(F) = -\left(\frac{a + \sum_{i=2}^{k} x_i}{x_1^2}\right) X_1 + \frac{1}{x_1}\left[\sum_{i=2}^{k} X_i\right]. \tag{13}$$

The proof will consist in checking that the choice of $\mathbf{X}_k$ given in the statement satisfies equations (12) and (13). The result is straightforward for $k = 3, 4, 5$ and we omit the details. So, from now on, we assume that $k \geqslant 6$.

We proceed in two steps:

First step: We will show that from expression (6) of X_1 as a seed of equations (12) we obtain the expressions of X_m for $m = 2, \ldots, k-1$ and X_k given by equations (7) and (8), respectively.

Second step: We will prove that the compatibility condition (13) is satisfied.

First step: We start with some preliminary observations:

Observation 1. Set $K_i := x_i + 1$ for $i = 1, \ldots, k-1$. Then $K_i(F) = x_{i+1} + 1$.

Observation 2. If we set $L_i := 1 + x_i + x_{i+1}$, then for all $1 \leqslant i \leqslant k-1$, $L_i(F) = 1 + x_{i+1} + x_{i+2} = L_{i+1}$, and

$$L_{k-1}(F) = 1 + x_k + x_{k+1} = 1 + x_k + \frac{a + \sum_{i=2}^{k} x_i}{x_1} = \frac{a + \sum_{i=1}^{k} x_i + x_1 x_k}{x_1}.$$

For this reason

(a) If $M_1 := \prod_{i=1}^{k-1}(1 + x_i + x_{i+1})$, then

$$M_1(F) = \frac{\left(a + \sum_{i=1}^{k} x_i + x_1 x_k\right)\left(\prod_{i=2}^{k-1}(1 + x_i + x_{i+1})\right)}{x_1}.$$

(b) Setting $M_m := \prod_{i=1, i\neq m-1, m}^{k-1}(1 + x_i + x_{i+1})$ for $2 \leqslant m \leqslant k-2$, we obtain

$$M_m(F) = \left(\prod_{i=2, i\neq m, m+1}^{k-1}(1 + x_i + x_{i+1})\right)\left(a + \sum_{i=1}^{k} x_i + x_1 x_k\right) \Big/ x_1.$$

(c) If $M_{k-1} := \prod_{i=1}^{k-3}(1 + x_i + x_{i+1})$, then $M_{k-1}(F) = \prod_{i=2}^{k-2}(1 + x_i + x_{i+1})$.

Observation 3. Set $N = a + \sum_{i=1}^{k-1} x_i - x_2 x_k$, then

$$N(F) = a + \sum_{i=2}^{k} x_i - x_3 x_{k+1} = a + \sum_{i=2}^{k} x_i - x_3 \frac{a + \sum_{i=2}^{k} x_i}{x_1}$$
$$= \frac{a + \sum_{i=2}^{k} x_i}{x_1}(x_1 - x_3) = x_{k+1}(x_1 - x_3).$$

Observation 4. Set $R = a + \sum_{i=1}^{k} x_i + x_1 x_k$, then

$$R(F) = a + \sum_{i=2}^{k} x_i + x_{k+1} + x_2 x_{k+1} = \frac{a + \sum_{i=2}^{k} x_i}{x_1}(1 + x_1 + x_2) = x_{k+1}(1 + x_1 + x_2).$$

Observation 5.

(a) For all $2 \leqslant i \leqslant k-2$ set $S_i = x_{i-1} - x_{i+1}$, then $S_i(F) = x_i - x_{i+2}$.

(b) Set $S_{k-1} = x_{k-2} - x_k$, then

$$S_{k-1}(F) = x_{k-1} - x_{k+1} = x_{k-1} - \frac{a + \sum_{i=2}^{k} x_i}{x_1} = -\frac{a + \sum_{i=2}^{k} x_i - x_1 x_{k-1}}{x_1}.$$

If we now consider the seed X_1 given by equation (6), using observations 1, 2a and 3 we obtain that

$$X_2 = \frac{(x_2 + 1)\left[\prod_{i=3}^{k-1}(1 + x_i + x_{i+1})\right]\left(a + \sum_{i=1}^{k} x_i + x_1 x_k\right)(x_1 - x_3)}{\prod_{i=1, i\neq 2}^{k} x_i}.$$

Now applying systematically observations 1, 2b, 4 and 5a, we obtain that for $2 \leqslant m \leqslant k-1$ the component $X_m = X_{m-1}(F)$ is given by equation (7).

Observe that in particular

$$X_{k-1} = X_{k-2}(F) = \frac{(x_{k-1}+1)\left[\prod_{i=1}^{k-3}(1+x_i+x_{i+1})\right]\left(a+\sum_{i=1}^{k}x_i+x_1x_k\right)(x_{k-2}-x_k)}{\prod_{i=1}^{k-2}x_i},$$

hence the term $L_{k-1} = 1 + x_{k-1} + x_k$ does not appear and so, in order to compute X_k we need to use observations 1, 2c, 4 and 5b, obtaining the expression of X_k given by (8).

Second step (compatibility condition (13)). A simple computations shows that

$$X_k(F) = \frac{\mathbf{A}}{x_1\left(\prod_{i=1}^{k}x_i\right)},$$

where

$$\mathbf{A} = -\left(a+\sum_{i=1}^{k}x_i\right)\left[\prod_{i=2}^{k-1}L_i\right]\left(a+\sum_{i=2}^{k}x_i+x_1\left(a+\sum_{i=3}^{k}x_i-x_2x_k\right)\right).$$

Another computation gives that

$$-\left(\frac{a+\sum_{i=2}^{k}x_i}{x_1^2}\right)X_1+\frac{1}{x_1}\left[\sum_{i=2}^{k}X_i\right] = \frac{\mathbf{B}}{x_1\left(\prod_{i=1}^{k}x_i\right)},$$

where

$$\mathbf{B} = -(x_1+1)\left(a+\sum_{i=2}^{k}x_i\right)\left[\prod_{i=2}^{k-1}L_i\right]\left(a+\sum_{i=1}^{k-1}x_i-x_2x_k\right)$$
$$+\left(a+\sum_{i=1}^{k}x_i+x_1x_k\right)\mathbf{C}-x_k(x_k+1)\left(a+\sum_{i=2}^{k}x_i-x_1x_{k-1}\right)\left[\prod_{i=1}^{k-2}L_i\right]$$

and

$$\mathbf{C} = \sum_{m=2}^{k-1}x_m(x_m+1)S_mM_m.$$

Recall that $L_i = 1 + x_i + x_{i+1}$ for all $i = 1,\ldots,k-1, S_m = x_{m-1} - x_{m+1}$ and $M_m = \prod_{i=1,i\neq m-1,m}^{k-1}(1+x_i+x_{i+1})$ for $2 \leqslant m \leqslant k-1$. Therefore we want to prove that $\mathbf{A} = \mathbf{B}$.

Step 2a. First we show that **B** contains L_2 and L_3 as a factors. To see this, it suffices to check that L_2 and L_3 are factors of **C**, which is an straightforward computation. In fact,

$$\mathbf{C} = L_2L_3Q_k(x_1,x_2,x_4,\ldots,x_k),$$

where

$$Q_k = -x_2(1+x_2)\left(\prod_{i=4}^{k-1}L_i\right)+(x_2-x_4)L_1\left(\prod_{i=4}^{k-1}L_i\right)+x_4(1+x_4)L_1\left(\prod_{i=5}^{k-1}L_i\right)$$
$$+\sum_{m=5}^{k-1}x_m(x_m+1)(x_{m-1}-x_{m+1})L_1\left(\prod_{i=4,j\neq m-1,m}^{k-1}L_j\right).$$

Observe that x_3 does not appear in the expression of Q_k. Hence L_2 and L_3 are factors in the expression of **B**, and then

$$\mathbf{B} = L_2L_3\left[-(x_1+1)\left(a+\sum_{i=2}^{k}x_i\right)\left[\prod_{i=4}^{k-1}L_i\right]\left(a+\sum_{i=1}^{k-1}x_i - x_2x_k\right)\right.$$
$$\left.+\left(a+\sum_{i=1}^{k}x_i + x_1x_k\right)Q_k - x_k(x_k+1)\left(a+\sum_{i=2}^{k}x_i - x_1x_{k-1}\right)L_1\left[\prod_{i=4}^{k-2}L_i\right]\right].$$

Step 2b. Now we state the following claim which will be proved at the end of the proof.

Claim: For $k \geqslant 6$, $Q_k(x_1, x_2, x_4, \ldots, x_k) = \left(\prod_{i=4}^{k-2} L_i\right)[x_1x_2L_{k-1} - x_{k-1}x_kL_1]$.

By using the claim, $\mathbf{B} = \left[\prod_{i=2}^{k-2} L_i\right]\mathbf{D}$, where

$$\mathbf{D} = -(x_1+1)\left(a+\sum_{i=2}^{k}x_i\right)\left(a+\sum_{i=1}^{k-1}x_i - x_2x_k\right)L_{k-1}$$
$$+\left(a+\sum_{i=1}^{k}x_i + x_1x_k\right)[x_1x_2L_{k-1} - x_{k-1}x_kL_1]$$
$$-x_k(x_k+1)\left(a+\sum_{i=2}^{k}x_i - x_1x_{k-1}\right)L_1.$$

Step 2c. Observe that

$$-x_k(x_k+1)\left(a+\sum_{i=2}^{k}x_i - x_1x_{k-1}\right)L_1 - x_{k-1}x_kL_1\left(a+\sum_{i=1}^{k}x_i + x_1x_k\right)$$
$$= -L_1x_kL_{k-1}\left(a+\sum_{i=2}^{k}x_i\right).$$

Thus **D** also contains the factor L_{k-1}, and therefore $\mathbf{B} = \left[\prod_{i=2}^{k-1} L_i\right]\mathbf{E}$, where

$$\mathbf{E} = \left[-(x_1+1)\left(a+\sum_{i=2}^{k}x_i\right)\left(a+\sum_{i=1}^{k-1}x_i - x_2x_k\right)\right.$$
$$\left.+\left(a+\sum_{i=1}^{k}x_i + x_1x_k\right)x_1x_2 - x_kL_1\left(a+\sum_{i=2}^{k}x_i\right)\right].$$

Step 2d. Is not difficult to check that **E** is a quadratic polynomial in x_3. Large but straightforward computations show that **E** vanishes either when $x_3 = x_{3,1} := -a - \sum_{i=1,i\neq 3}^{k} x_i$ (that is when $a + \sum_{i=1}^{k} x_i = 0$), and when

$$x_3 = x_{3,2} := -\left(a+\sum_{i=4}^{k-1}x_i + \frac{x_2(1-x_1x_k)}{1+x_1}\right).$$

In summary, as a quadratic polynomial in x_3, **E** factorizes as

$$\mathbf{E} = -(x_1+1)\left(x_3+a+\sum_{i=1,i\neq 3}^{k}x_i\right)\left(x_3+a+\sum_{i=4}^{k-1}x_i + (x_2(1-x_1x_k))/(x_1+1)\right)$$
$$= -\left(a+\sum_{i=1}^{k}x_i\right)\left(a+\sum_{i=2}^{k-1}x_i + x_1\left(a+\sum_{i=3}^{k-1}x_i - x_2x_k\right)\right).$$

So, finally, we get that

$$\mathbf{B} = -\left[\prod_{i=2}^{k-1} L_i\right]\left(a + \sum_{i=1}^{k} x_i\right)\left(a + \sum_{i=2}^{k-1} x_i + x_1\left(a + \sum_{i=3}^{k-1} x_i - x_2x_k\right)\right) = \mathbf{A},$$

as we wanted to show.

To end the proof it only remains to prove the claim. We proceed by induction. That it is true when $k = 6$ is straightforward. Assume now that the claim is true for Q_k, then

$$\begin{aligned} Q_{k+1} &= -x_2(1+x_2)\left(\prod_{i=4}^{k} L_i\right) + (x_2 - x_4)L_1\left(\prod_{i=4}^{k} L_i\right) + x_4(1+x_4)L_1\left(\prod_{i=5}^{k} L_i\right) \\ &\quad + \sum_{m=5}^{k-1} x_m(1+x_m)(x_{m-1} - x_{m+1})L_1\left(\prod_{i=4, i\neq m-1,m}^{k} L_i\right) \\ &= L_kQ_k + L_1\left(\prod_{i=4}^{k-2} L_i\right)(x_k(1+x_k)(x_{k-1} - x_{k+1})). \end{aligned}$$

By using the hypothesis of induction, we obtain that

$$\begin{aligned} Q_{k+1} &= L_k\left(\prod_{i=4}^{k-2} L_i\right)[x_1x_2L_{k-1} - x_{k-1}x_kL_1] \\ &\quad + L_1\left(\prod_{i=4}^{k-2} L_i\right)(x_k(1+x_k)(x_{k-1} - x_{k+1})) \\ &= \left(\prod_{i=4}^{k-2} L_i\right)(x_1x_2L_kL_{k-1} + L_1[x_k(1+x_k)(x_{k-1} - x_{k+1}) - x_{k-1}x_kL_k]). \end{aligned}$$

Thus

$$\begin{aligned} Q_{k+1} &= L_k\left(\prod_{i=4}^{k-2} L_i\right)[x_1x_2L_{k-1} - x_{k-1}x_kL_1] + L_1\left(\prod_{i=4}^{k-2} L_i\right)(x_k(1+x_k)(x_{k-1} - x_{k+1})) \\ &= \left(\prod_{i=4}^{k-2} L_i\right)(x_1x_2L_kL_{k-1} + L_1[x_k(1+x_k)(x_{k-1} - x_{k+1}) - x_{k-1}x_kL_k]). \end{aligned}$$

An easy computation shows that $x_k(1+x_k)(x_{k-1} - x_{k+1}) - x_{k-1}x_kL_k = -L_{k-1}x_kx_{k+1}$, and the result follows. Therefore, theorem 1 is proved.

3. Geometrical issues in the odd case

Before proving theorem 2 and propositions 3 and 4 we need some preliminary results. Recall that a map H which is a first integral for $G^p := G \circ \overset{p)}{\cdots} \circ G$ is also called sometimes *a p-first integral* or simply, for short, a *p-integral*, of G, see [31].

The following lemma, which is very easy to prove, gives light on one utility of p-first integrals, specially if they are not symmetric functions of their arguments.

Lemma 5. *Let H be a p-integral of a map G. Then for any symmetric function of p variables S, the function $V_S := S(H, H \circ G, H \circ G^2, \ldots, H \circ G^{p-1})$ is a first integral of G.*

Lemma 6. *Set $k = 2\ell + 1$, $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathcal{Q}^+$ and let W be the function given in (10). Then, if we define the polynomial*

$$Z(\mathbf{x}) := \left(\prod_{i=1}^{k} x_i\right)[W(\mathbf{x}) - W(F(\mathbf{x}))]$$
$$= \prod_{j=0}^{\ell} x_{2j+1}(x_{2j+1} + 1) - \left(a + \sum_{i=1}^{2\ell+1} x_i\right)\prod_{j=1}^{\ell} x_{2j}(x_{2j} + 1),$$

it holds that $Z(F(\mathbf{x})) = \det(DF(\mathbf{x}))Z(\mathbf{x})$.

Proof. In theorem 2 (a) we will prove that W is a 2-integral of F. Thus, if we define $\tilde{Z} := W - W(F)$, then

$$\tilde{Z}(F(\mathbf{x})) = -\tilde{Z}(\mathbf{x}). \qquad (14)$$

From the above equality it is clear that $\{\mathbf{x} \in \mathcal{Q}^+ : \tilde{Z}(\mathbf{x}) = 0\} = \{\mathbf{x} \in \mathcal{Q}^+ : Z(\mathbf{x}) = 0\}$ is an invariant hypersurface by F. Note also that if we define $\Pi(\mathbf{x}) := \prod_{j=1}^{k} x_j$, it holds that

$$\Pi(F(\mathbf{x})) = \frac{a + \sum_{i=2}^{k} x_i}{x_1^2}\Pi(\mathbf{x}) = -\det(DF(\mathbf{x}))\Pi(\mathbf{x}). \qquad (15)$$

Since $Z(\mathbf{x}) = \Pi(\mathbf{x})\tilde{Z}(\mathbf{x})$, by using equalities (14) and (15), we obtain that

$$Z(F(\mathbf{x})) = \Pi(F(\mathbf{x}))\tilde{Z}(F(\mathbf{x})) = \det(DF(\mathbf{x}))\Pi(\mathbf{x})\tilde{Z}(\mathbf{x}) = \det(DF(\mathbf{x}))Z(\mathbf{x}),$$

as we wanted to see. □

Proof of theorem 2. (a) The proof of the equality $W(F^2(\mathbf{x})) = W(\mathbf{x})$ is straightforward. We have obtained expression (10) inspired in the results of [26]. In that paper it is proved that

$$\frac{(y_n + 1)(y_{n+2} + 1)}{y_{n+1}} = \frac{(y_{n+2} + 1)(y_{n+4} + 1)}{y_{n+3}},$$

where $\{y_n\}$ is the sequence given by the third-order Lyness' recurrence $y_{n+3} = (a + y_{n+1} + y_{n+2})/y_n$. Note that this property is equivalent to say that for $k = 3$, W is a 2-integral for F.

(b–c) By applying lemma 5 with $S(u, v) = u + v$ and $S(u, v) = uv$, we obtain the first integrals V_3 and V_1, respectively. The functionally independence of V_1, V_2 and V_3, for $\ell \geqslant 2$, follows from straightforward computations and it is already established in [13].

(d–f) From lemma 6 we know that $Z(F(\mathbf{x})) = \det(DF(\mathbf{x}))Z(\mathbf{x})$, where recall that $Z(\mathbf{x}) = \left(\prod_{i=1}^{k} x_i\right)[W(\mathbf{x}) - W(F(\mathbf{x}))]$. Note also that

$$\det(DF(\mathbf{x})) = (-1)^k \frac{a + x_2 + \cdots + x_{k-1}}{x_k^2}.$$

Since when k is odd $\det(DF) < 0$ on $\mathcal{Q}^+$, equation $Z(F) = \det(DF)Z$ means that

$$\mathcal{G} = \{\mathbf{x} \in \mathcal{Q}^+ : W(\mathbf{x}) = W(F(\mathbf{x}))\} = \{\mathbf{x} \in \mathcal{Q}^+ : Z(\mathbf{x}) = 0\}$$

is invariant by F and that F maps the region $\{\mathbf{x} : Z(\mathbf{x}) > 0\}$ into the region $\{\mathbf{x} : Z(\mathbf{x}) < 0\}$ and vice versa. Furthermore it implies that the dynamics of F^2 on each of these sets are conjugated, being the map F itself the conjugation. Moreover, any periodic orbit with odd period must lie in $\mathcal{G}$, as we wanted to see.

(g–h) By using the change of variables theorem it is easy to see that if G is a diffeomorphism of $\mathcal{U}$, and on this region μ is a positive function that satisfies $\mu(G(\mathbf{x})) = \det(DG(\mathbf{x}))\mu(\mathbf{x})$, then

$$m(B) = \int_B \frac{1}{\mu(\mathbf{x})}\,\mathrm{d}\mathbf{x}$$

is an invariant measure for G. By lemma 6 we know that $Z(F(\mathbf{x})) = \det(DF(\mathbf{x}))Z(\mathbf{x})$ and by equality (15), that $\Pi(F(\mathbf{x})) = -\det(DF(\mathbf{x}))\Pi(\mathbf{x})$. By using these results we have that $Z(F^2(\mathbf{x})) = \det(DF^2(\mathbf{x}))Z(\mathbf{x})$ and $\Pi(F^2(\mathbf{x})) = \det(DF^2(\mathbf{x}))\Pi(\mathbf{x})$, being both equalities in the corresponding domains, which are invariant by F^2. Hence (f) and (g) follow. □

3.1. Order reduction

Recall that in the previous section we have seen that when $k = 2\ell + 1$, the regions $\{\mathbf{x} : Z(\mathbf{x}) > 0\}$ and $\{\mathbf{x} : Z(\mathbf{x}) < 0\}$ are invariant by F^2, and the dynamics on both region are conjugated. This observation allows us to give a new application of the invariant W. We can reduce the study of the dynamics of F on $\{\mathbf{x} : Z(\mathbf{x}) \neq 0\}$ to the study of a new $(k-1)$-dimensional map, having one more parameter.

For instance for $n = 3$, we get that $W(x, y, z) = (x+1)(z+1)/y$, and hence any admissible level surface $\{\mathbf{x} : W(\mathbf{x}) = w\}, w \neq 0$, can be described as $y = k(x+1)(z+1)$, where $k = 1/w$. Therefore

$$F^2|_{\{W=w\}}(x, y, z) = \left(z, \frac{a+z+k(x+1)(z+1)}{x}, \frac{a+k+z(k+1)}{kx(z+1)}\right),$$

and we can reduce the study of the dynamics of F^2 to the study of the reduced map

$$\tilde{F}_2(x, z) = \left(z, \frac{a+k+z(k+1)}{kx(z+1)}\right),$$

or, equivalently, the study of the second-order difference equation

$$y_{n+2} = \frac{a+k+y_{n+1}(k+1)}{ky_n(y_{n+1}+1)}, \tag{16}$$

as in [26, equations (8) and (9)]. The dynamics of this difference equation is studied in [10, example 3]. Equation (16) is sometimes called *generalized Lyness' recurrence*, see [24].

For $n = 5$, the integral of F^2 is $W(x, y, z, t, s) = (x+1)(z+1)(s+1)/(yt)$. Again any admissible level surface $\{\mathbf{x} : W(\mathbf{x}) = w\}, w \neq 0$, can be described as $t = k(x+1)(z+1)(s+1)/y$, where $k = 1/w$. Therefore proceeding as before we can reduce the study of the dynamics of F^2 to the study of the reduced map

$$\tilde{F}_2(x, y, z, s) = \left(y, z, t, \frac{p_2(z, s; k)x^2 + p_1(y, z, s; a, k)x + p_0(y, z, s; a, k)}{xy^2}\right),$$

where $p_2(z, s; k) = k(s+1)(z+1)$, $p_1(y, z, s; a, k) = 2k(s+1)(z+1) + y(a+s+z)$, and $p_0(y, z, s; a, k) = k(s+1)(z+1) + y(z+s+a+y)$, or equivalently to the difference equation

$$y_{n+4} = \frac{p_2(y_{n+2}, y_{n+3}; k)y_n^2 + p_1(y_{n+1}, y_{n+2}, y_{n+3}; a, k)y_n + p_0(y_{n+1}, y_{n+2}, y_{n+3}; a, k)}{y_n {y_{n+1}}^2}.$$

Clearly, the described procedure can be generalized to higher dimensions.

4. Dynamics of the low-dimensional cases

This subsection is devoted to prove propositions 3 and 4.

Along the section we will use a straightforward consequence of the following result:

Theorem 7 ([4]). *Let $\bar{\mathbf{x}}$ be the fixed point of F in $\mathcal{Q}^+$. For any $h > V_1(\bar{\mathbf{x}})$, the level sets $\{V_1 = h\} \cap \mathcal{Q}^+$ are homeomorphic to $\mathbb{S}^{k-1}$.*

Corollary 8. *Let $K \neq \emptyset$ be by the intersection of some level sets of different first integrals of F, including V_1 among them. Then $K \cap \mathcal{Q}^+$ is a compact set, invariant by F.*

For the sake of completeness and to compare with the cases $k = 4, 5$, we start by recalling some results for the case $k = 3$, most of them already proved in [9].

4.1. The three-dimensional map

For $k = 3$,

$$F(x, y, z) = \left(y, z, \frac{a + y + z}{x}\right),$$

the Lie symmetry given in theorem 1, is

$$\mathbf{X}_3 := \left[x(x+1)(1+y+z)(a+x+y-yz)\frac{\partial}{\partial x} + y(y+1)(x-z)(a+x+y+z+xz)\frac{\partial}{\partial y}\right.$$
$$\left. + z(z+1)(1+x+y)(a+y+z-xy)\frac{\partial}{\partial z}\right] \Big/ (xyz)$$

and since $\mathbf{X}_3(V_i) = 0$, for $i = 1, 2$, the functions V_1 and V_2 given in (3) and (4) are first integrals for $\mathbf{X}_3$ and F. Also, by theorem 2, $W(x, y, z) := (x+1)(z+1)/y$ is a 2-integral of F; $\mathcal{G} = \{\mathbf{x} \in \mathcal{Q}^+ : Z(\mathbf{x}) = 0\}$ is invariant by F, where

$$Z(\mathbf{x}) = x(x+1)z(z+1) - (a+x+y+z)y(y+1);$$

and F maps $\mathcal{G}^+ := \{\mathbf{x} \in \mathcal{Q}^+ : Z(\mathbf{x}) > 0\}$ into $\mathcal{G}^- := \{\mathbf{x} \in \mathcal{Q}^+ : Z(\mathbf{x}) < 0\}$ and vice versa.

Let $\bar{\mathbf{x}}$ be the fixed point in $\mathcal{Q}^+$, of F. Set $h > V_1(\bar{\mathbf{x}}), k > V_2(\bar{\mathbf{x}})$. In [9] it is proved that $\{V_1 = h\} \cap \mathcal{Q}^+$ and $\{V_2 = k\} \cap \mathcal{Q}^+$ are diffeomorphic to two-dimensional spheres, and their transversal intersections are formed by exactly two disjoints curves, both diffeomorphic to circles. Their non-transversal intersections correspond to:

(a) The 2-periodic points of F (which are equilibrium points of $\mathbf{X}_3$) given by the curve $\mathcal{L} := \{(x, (x+a)/(x-1), x) | x > 1\}$..
(b) The levels placed at $\mathcal{G}$. Those placed at $\mathcal{G} \backslash \{\bar{\mathbf{x}}\}$ are formed by exactly one curve, diffeomorphic to a circle.

Finally, since $\mathbf{X}_3$ is also a Lie symmetry of F^2, as a consequence of [9, theorem 18] or [10, theorem 1], we know that the map F^2 restricted to each of the sets $\{V_1 = h\} \cap \{V_2 = k\}$, which is not simply a fixed point of F^2, is conjugated to a rotation. Further discussion about the possible rotation numbers can be found in [9].

4.2. The four-dimensional map

Proof of proposition 3. Equations (6)–(8) give the following Lie symmetry for the four-dimensional Lyness' map:

$$\mathbf{X}_4 = \left[x(x+1)(1+y+z)(1+z+t)(a+x+y+z-yt)\frac{\partial}{\partial x}\right.$$
$$+ y(y+1)(1+z+t)(a+x+y+z+t+xt)(x-z)\frac{\partial}{\partial y}$$
$$+ z(z+1)(1+x+y)(a+x+y+z+t+xt)(y-t)\frac{\partial}{\partial z}$$
$$\left. - t(t+1)(1+x+y)(1+y+z)(a+y+z+t-xz)\frac{\partial}{\partial t}\right] \Big/ (xyzt).$$

A straightforward computation shows that the Lie symmetry satisfies $\mathbf{X}_4(V_i) = 0$, for $i = 1, 2$. Hence, the orbits of both F and $\mathbf{X}_4$ generically lie in a two-dimensional surface of the form $I_{h,k} := \{V_1 = h\} \cap \{V_2 = k\} \cap \mathcal{Q}^+$.

Let $C_{h,k}$ be a connected component of $I_{h,k}$. From corollary 8 we know that $C_{h,k}$ is compact. If $\{V_1 = h\}$ and $\{V_2 = k\}$ intersect transversally on $C_{h,k}$, then for all points

$$\operatorname{Rank}\begin{pmatrix} (V_1)_x & (V_1)_y & (V_1)_z & (V_1)_t \\ (V_2)_x & (V_2)_y & (V_2)_z & (V_2)_t \end{pmatrix} = 2.$$

This fact implies that the dual 2-form associated with the 2-field

$$\begin{aligned}\nabla V_1 \wedge \nabla V_2 = {} & [(V_1)_x(V_2)_y - (V_1)_y(V_2)_x]\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + [(V_1)_x(V_2)_z - (V_1)_z(V_2)_x]\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} \\ & + [(V_1)_x(V_2)_t - (V_1)_t(V_2)_x]\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t} + [(V_1)_y(V_2)_z - (V_1)_z(V_2)_y]\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} \\ & + [(V_1)_y(V_2)_t - (V_1)_t(V_2)_y]\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t} + [(V_1)_z(V_2)_t - (V_1)_t(V_2)_z]\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t},\end{aligned}$$

given by

$$\begin{aligned}\omega = {} & [(V_1)_z(V_2)_t - (V_1)_t(V_2)_z]\,\mathrm{d}x\,\mathrm{d}y + [(V_1)_z(V_2)_x - (V_1)_x(V_2)_z]\,\mathrm{d}x\,\mathrm{d}z \\ & + [(V_1)_z(V_2)_y - (V_1)_y(V_2)_z]\,\mathrm{d}x\,\mathrm{d}t + [(V_1)_t(V_2)_x - (V_1)_x(V_2)_t]\,\mathrm{d}y\,\mathrm{d}z \\ & + [(V_1)_t(V_2)_y - (V_1)_y(V_2)_t]\,\mathrm{d}y\,\mathrm{d}t + [(V_1)_x(V_2)_y - (V_1)_y(V_2)_x]\,\mathrm{d}z\,\mathrm{d}t\end{aligned}$$

is nonzero at every point of $C_{h,k}$, and therefore it is orientable, see [1, section 2.5] or [25, section 10.1].

Let $\mathbf{X}_4^{h,k}$ denote the restriction of the vector field $\mathbf{X}_4$ to the invariant surface $C_{h,k}$. Some computations show that the unique equilibrium point of $\mathbf{X}_4$ in $\mathcal{Q}^+$ is the fixed point of F. Hence $\mathbf{X}_4^{h,k}$ has no equilibrium points, and therefore the Poincaré–Hopf formula gives

$$0 = i\left(\mathbf{X}_4^{h,k}\right) = \chi(C_{h,k}) = 2 - 2g$$

where $i\left(\mathbf{X}_4^{h,k}\right)$ denotes the sum of the indices of the equilibrium points of $\mathbf{X}_4^{h,k}$ in $C_{h,k}$ and $\chi(C_{h,k})$ and g are the Euler characteristic and the genus of the surface $C_{h,k}$, respectively. Hence the genus of $C_{h,k}$ is one. An orientable, compact, connected surface of genus one is a torus, as we wanted to prove. □

4.3. The five-dimensional map

Proof of proposition 4. By theorem 2 we know that

$$W(x, y, z, t, s) := \frac{(x+1)(z+1)(s+1)}{yt}$$

is a 2-integral of F. Moreover, the three functionally independent first integrals of F given in (3), (4) and (11) are

$$\begin{aligned} V_1(x, y, z, t, s) &= \frac{(a+x+y+z+t+s)(x+1)(y+1)(z+1)(t+1)(s+1)}{xyzts}, \\ V_2(x, y, z, t, s) &= \frac{(a+x+y+z+t+s+xs)(1+x+y)(1+y+z)(1+z+t)(1+t+s)}{xyzts}, \\ V_3(x, y, z, t, s) &= \frac{x(x+1)z(z+1)s(s+1) + (a+x+y+z+t+s)y(y+1)t(t+1)}{xyzts}. \end{aligned}$$

Some tedious computations show that the hypersurface $\mathcal{G}$ is in the locus of non-transversality of the three level sets of the integrals V_i, $i = 1, 2, 3$ in $\mathcal{Q}^+$. Recall that precisely, $\mathcal{G} = \{\mathbf{x} \in \mathcal{Q}^+ : Z(\mathbf{x}) = 0\}$ is invariant by F, where

$$Z(\mathbf{x}) = x(x+1)z(z+1)s(s+1) - (a+x+y+z+t+s)y(y+1)t(t+1),$$

and that F maps $\mathcal{G}^+ = \{\mathbf{x} \in \mathcal{Q}^+ : Z(\mathbf{x}) > 0\}$ into $\mathcal{G}^- = \{\mathbf{x} \in \mathcal{Q}^+ : Z(\mathbf{x}) < 0\}$ and vice versa.

Equations (6)–(8) give the following Lie symmetry for the five-dimensional Lyness' map:

$$\begin{aligned}\mathbf{X}_5 = \Big[& x(x+1)(1+y+z)(1+z+t)(1+t+s)(a+x+y+z+t-ys)\frac{\partial}{\partial x}\\ & + y(y+1)(1+t+s)(1+z+t)(a+x+y+z+t+s+xs)(x-z)\frac{\partial}{\partial y}\\ & + z(z+1)(1+x+y)(1+t+s)(a+x+y+z+t+s+xs)(y-t)\frac{\partial}{\partial z}\\ & + t(t+1)(1+x+y)(1+y+z)(a+x+y+z+t+s+xs)(z-s)\frac{\partial}{\partial t}\\ & - s(s+1)(1+x+y)(1+y+z)(1+z+t)(a+y+z+t+s-tx)\frac{\partial}{\partial s}\Big] \Big/ (xyzts).\end{aligned}$$

Again, direct computations show that $\mathbf{X}_5(V_i) = 0$, for $i = 1, 2, 3$. Hence the orbits of both F, and $\mathbf{X}_5$ lie in a two-dimensional surface of the form $I_{h,k,\ell} := \{V_1 = h\} \cap \{V_2 = k\} \cap \{V_3 = \ell\} \cap \mathcal{Q}^+$.

Let $C_{h,k,\ell}$ be a connected component of $I_{h,k,\ell}$. From corollary 8 we know that $C_{h,k,\ell}$ is compact. If $\{V_1 = h\}, \{V_2 = k\}$ and $\{V_3 = k\}$ intersect transversally on $C_{h,k,\ell}$, then for all points

$$\mathrm{Rank}\begin{pmatrix} (V_1)_x & (V_1)_y & (V_1)_z & (V_1)_t & (V_1)_s \\ (V_2)_x & (V_2)_y & (V_2)_z & (V_2)_t & (V_2)_s \\ (V_3)_x & (V_3)_y & (V_3)_z & (V_3)_t & (V_3)_s \end{pmatrix} = 3.$$

Similarly than in the case $k = 4$, this fact implies that the dual 2-form associated with the 3-field $\nabla V_1 \wedge \nabla V_2 \wedge \nabla V_3$ is nonzero at every point of $C_{h,k,\ell}$, and therefore this set is a two-dimensional orientable manifold.

It is not difficult to check that all the equilibrium points of $\mathbf{X}_5$ in $\mathcal{Q}^+$ are the points of the curve

$$\mathcal{L} = \left\{ \mathbf{x} = \left(x, \frac{2x+a}{x-2}, x, \frac{2x+a}{x-2}, x \right) \text{ with } x > 2 \right\},$$

which contains a continuum of two periodic points and the fixed point of F. Moreover $\mathcal{L}$ belongs to the locus of non-transversality of the integrals V_1, and V_2 in $\mathcal{Q}^+$.

The above observation implies that $\mathbf{X}_5$ has no equilibrium points in any level set $I_{h,k,\ell}$ where the three first integrals intersect transversally. Therefore the Poincaré–Hopf formula gives

$$0 = i\big(\mathbf{X}_5^{h,k,\ell}\big) = \chi(C_{h,k,\ell}) = 2 - 2g,$$

for each connected component $C_{h,k,\ell}$ of such a level set (where $\mathbf{X}_5^{h,k,\ell}$ is the restriction of $\mathbf{X}_5$ to $C_{h,k,\ell}$).

Hence $g = 1$, which implies that $C_{h,k,\ell}$ is a torus (since it is two dimensional, orientable, compact, manifold of genus one), as we wanted to proof.

Finally, observe that the fact that F maps $\mathcal{G}^+$ into $\mathcal{G}^-$, and vice versa, implies that most $I_{h,k,\ell}$ have at least one connected component on each set. In fact, it seems that each $I_{h,k,\ell}$, not

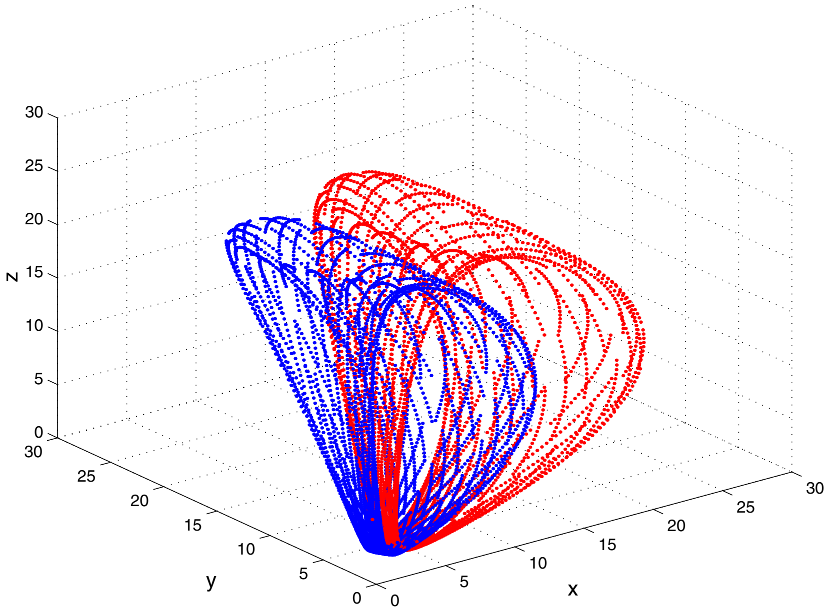

Figure 3. Projection into $\mathbb{R}^3$ of the first 10^4 iterates of the Lyness' map, for $k = 5$ and $a = 4$, starting at $(1, 2, 3, 4, 5)$. Odd and even iterates are in different connected components.

included in Γ, has exactly two connected components in $\mathcal{Q}^+$, as it happens when $k = 3$. See figures 2 and 3 for two illustrations of this assertion. □

From the above proof it is clear that if some $I_{h,k,\ell}$ cuts $\mathcal{L}$ then the first three integrals do not cut transversally on it. Let us see that in general $I_{h,k,\ell} \cap \mathcal{L} = \emptyset$. This will be a consequence of the shape of the function

$$v_1(x) := V_1\left(x, \frac{2x+a}{x-2}, x, \frac{2x+a}{x-2}, x\right), \qquad x > 2, a \geqslant 0.$$

This function has a global minimum at the coordinate given by the fixed point $x = 2 + \sqrt{4+a}$, and $\lim_{x\to 2^+} v_1(x) = \lim_{x\to+\infty} v_1(x) = +\infty$.

Thus, given $h > v_1(2+\sqrt{4+a})$ there are only two solutions $x_1(h) < 2+\sqrt{4+a} < x_2(h)$ of the equation $v_1(x) = h$. Now set

$$v_i(x) := V_i\left(x, \frac{2x+a}{x-2}, x, \frac{2x+a}{x-2}, x\right), \qquad x > 2, a \geqslant 0,$$

for $i = 2, 3$. Then, for any value of k and ℓ satisfying $k \notin \{v_2(x_1(h)), v_2(x_2(h))\}$ or $\ell \notin \{v_3(x_1(h)), v_3(x_2(h))\}$, we obtain that $I_{h,k,\ell}$, does not intersect $\mathcal{L}$.

5. Conclusions

Several properties for the k-dimensional Lyness' map F have been given, like the existence of a Lie symmetry for F and of a new and simple invariant for F^2. This Lie symmetry together with the new invariant give information for $k = 4$ and 5 about the dynamics and the topology of the level surfaces where the dynamics of F is confined. Some general results for k odd have been also presented. However, on the contrary that happens in the cases $k = 2$ and 3, the numerical explorations indicate that for $k = 4, 5$ the orbits of the map are not contained in the orbits of the flow of the Lie symmetry *with the same initial condition*, although they are placed in the same manifold, which has dimension $k - E\left(\frac{k+1}{2}\right)$. This is an obstruction to apply the theoretical tools developed in [10].

Numerical simulations seem to show that for some initial conditions the projection in $\mathbb{R}^3$ of the iterates of F when $k = 6, 7$ fill densely a two-dimensional manifold, indicating that probably they live in a three-dimensional manifold of $\mathbb{R}^6$ and $\mathbb{R}^7$. These facts are coherent with the conjecture of [13] about the number of independent first integrals of the Lyness' maps, and show that for $k \geqslant 6$ the dynamics are much more complicated.

When $k = 2\ell$, the simplest scenario that we imagine for the dynamics of the k-dimensional Lyness' map is that most of the orbits lie on invariant manifolds which are diffeomorphic to ℓ-dimensional tori, $S^1 \times \overset{\ell)}{\cdots} \times S^1$. On the other hand, when $k = 2\ell + 1$, most of them lie on two diffeomorphic copies of $S^1 \times \overset{\ell)}{\cdots} \times S^1$, separated by the invariant set $\mathcal{G}$. Moreover these orbits jump from one of these tori to the other one and vice versa.

In any case, much more research must be done in order to have a total understanding of the dynamics and the geometrical structure of high-dimensional Lyness' maps.

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